# Time-delay inteferometry for LISA: IInd generation TDI

Sanjeev Dhurandhar

IUCAA

Pune

Collaborators: J-Y Vinet and R. Nayak

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# Plan of Talk

- Introduction: LISA
- Laser frequency noise cancellation problem
- The TDI problem for LISA: Ist, modified Ist and IInd generation TDI
- Mathematical structure: module of syzygies
- IInd generation TDI for LISA with one arm dysfunctional

# Introduction: LISA

- Ground-based Laser Inteferometric Detector Network:
  Detection frequency range: 10 Hz to KHz.

- Ground-based gravitational wave detectors are entering the advanced detector stage

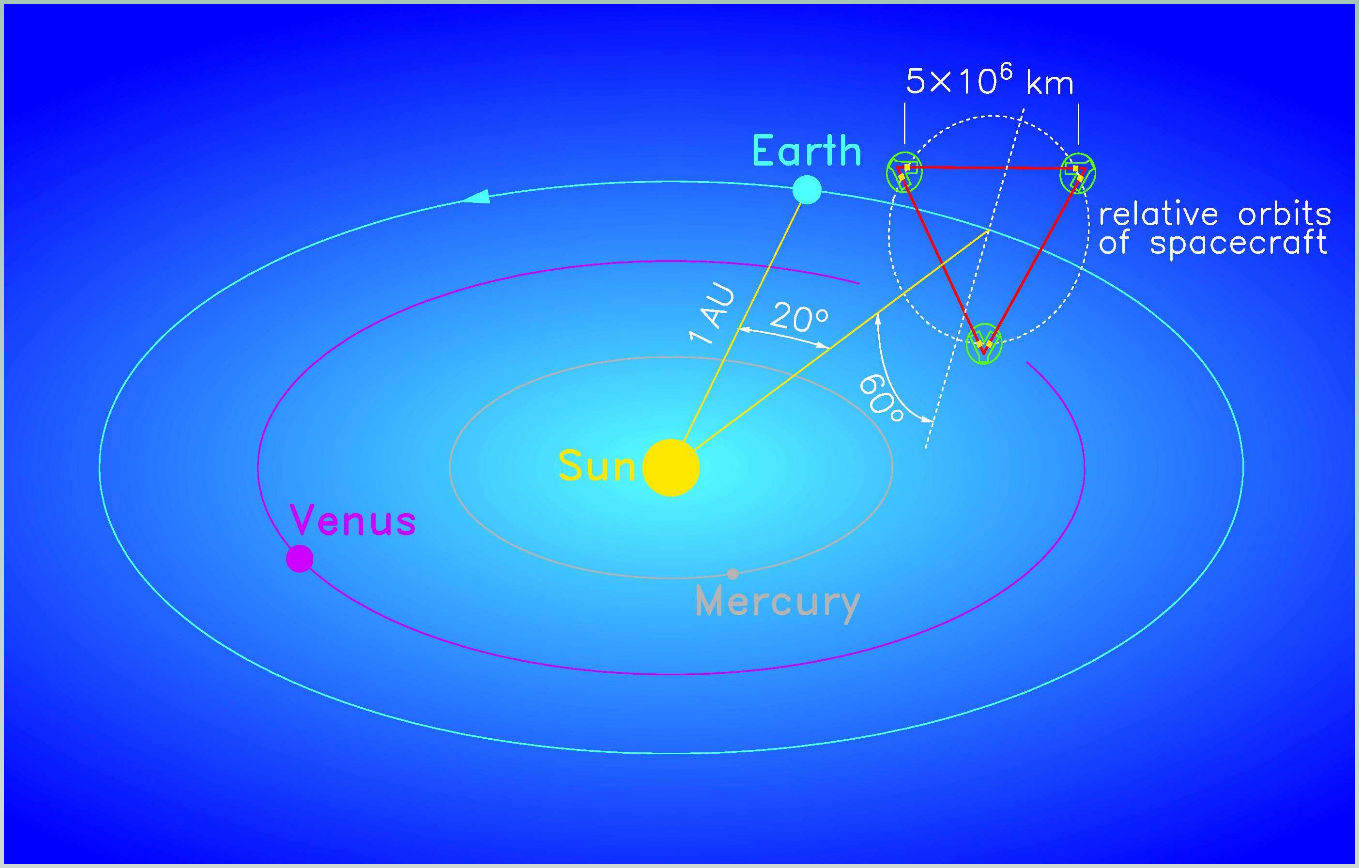
- The Space Antenna LISA: Why go to Space?

  – Low frequency searches :$10^{-5}$ Hz to $10^{-1}$ Hz
    Complementary to groundbased detectors - just as the different astronomies, optical, radio, etc. complement each other.

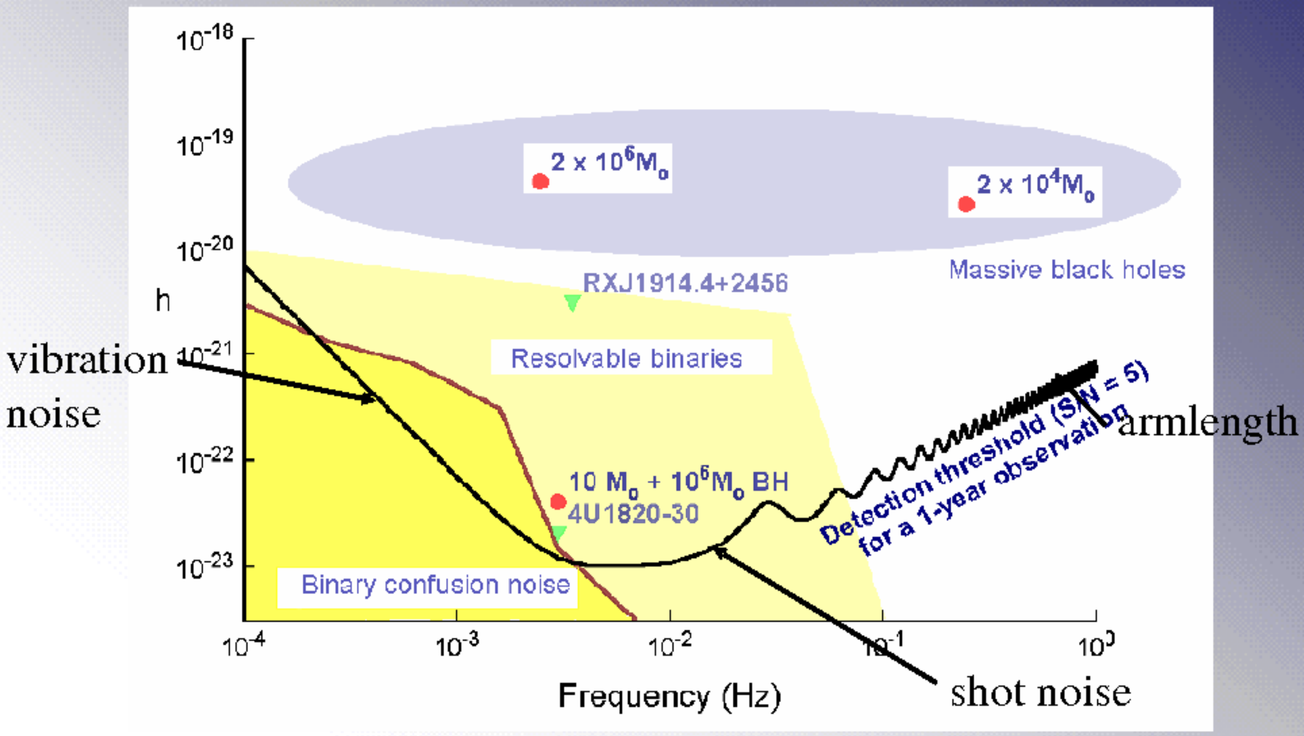
  – Gauranteed Gravitational Wave Sources

- The LISA Project: ESA & NASA

# The LISA Project



# LISA noise curve



# Laser Frequency Noise

- Frequency of Laser: $\nu_0 \sim 3 \times 10^{14}$ Hz
- Frequency noise: $\widetilde{\Delta\nu} \sim 10$ Hz / $\sqrt{\text{Hz}}$
- $h_{noise} \ = \ C(t) \ \equiv \ \frac{\Delta\nu(t)}{\nu_0} \sim 3 \times 10^{-14}$
- But $h_{sens} \sim 10^{-21}, 10^{-22}$
- **7 or 8 orders of magnitude**

# Cancelling laser frequency noise in an unequal arm Interferometer

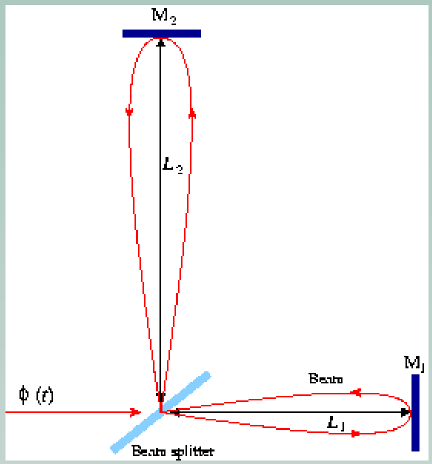
Equal arms: This noise is automatically cancelled

$C(t)$: Laser frequency noise

$$C_1(t) = C(t-2L_1) - C(t) = (\mathcal{D}_1^2 - 1)C(t)$$

$$C_2(t) = C(t-2L_2) - C(t) = (\mathcal{D}_2^2 - 1)C(t)$$

$$\Delta C(t) = C_1(t) - C_2(t) \neq 0$$

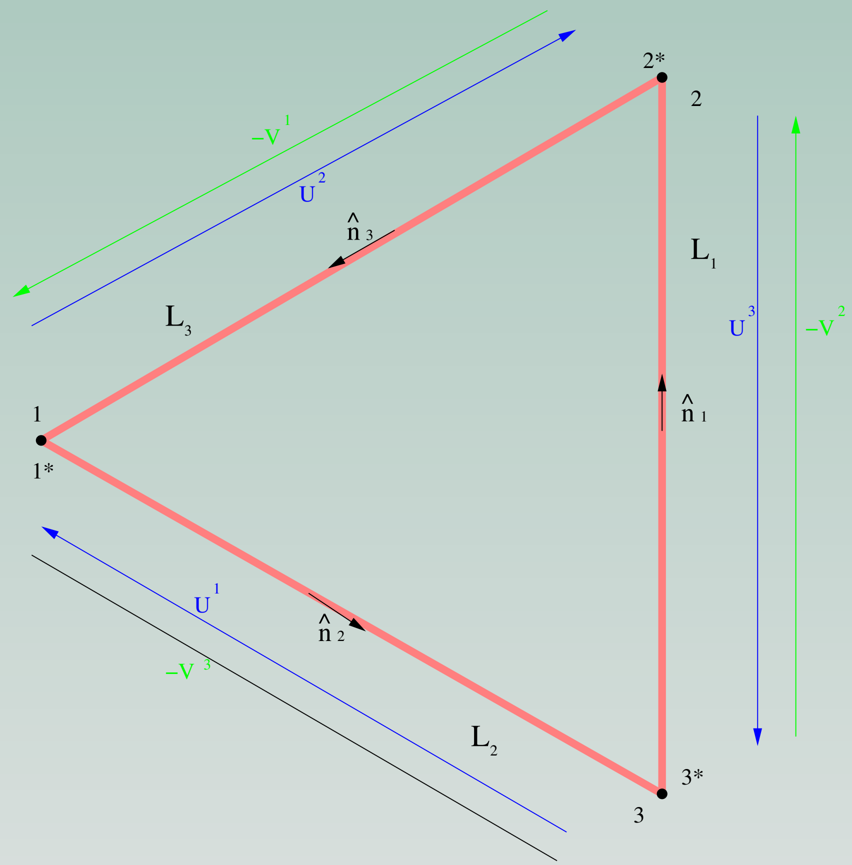



$$\begin{aligned} X(t) &= [C_1(t-2L_2) - C_1(t)] - [C_2(t-2L_1) - C_2(t)] \\ &= [(\mathcal{D}_2^2 - 1)(\mathcal{D}_1^2 - 1) - (\mathcal{D}_1^2 - 1)(\mathcal{D}_2^2 - 1)]C(t) \\ &= 0 \end{aligned}$$

**Just LCM! Commutative algebra! Equal pathlengths**

# Schematic LISA

LISA: Interferometric triangle with unequal arms




- All $L_i$ unequal
- $L_i \sim 5 \times 10^6$ km $\sim 17$ sec.
- Six elementary data streams:

  $$U^1, U^2, U^3, V^1, V^2, V^3$$

- Example:

  $$U^1 : 3 \longrightarrow 1$$

  $$-V^1 : 2 \longrightarrow 1$$

# The six elementary data streams

In LISA the data are recorded as fractional Doppler shifts

Three laser frequency noises: $C_i(t) = \frac{\Delta\nu_i(t)}{\nu_0}$

For example: $U^1(t) = C_3(t - L_2(t)) - C_1(t) + U^1_{\rm GW} + U^1_{\rm opt} + U^1_{\rm pf}$

Algebra of finite difference operators $x, y, z; l, m, n$

$x : 1 \longrightarrow 2, \;\; l : 2 \longrightarrow 1$, ... + cyclic permutations

Displaying only the laser frequency terms: $U^1(t) = zC_3(t) - C_1(t) + \ldots$

$$\begin{aligned} U^1 &= zC_3 - C_1 + \ldots & V^1 &= C_1 - lC_2 + \ldots \\ U^2 &= xC_1 - C_2 + \ldots & V^2 &= C_2 - mC_3 + \ldots \\ U^3 &= yC_2 - C_3 + \ldots & V^3 &= C_3 - nC_1 + \ldots \end{aligned}$$

# The General TDI for LISA

A general data combination:

$$X = \sum_{i=1}^{3} p_i V^i + q_i U^i$$

where $p_i$ and $q_i$ are the polynomials in $x, y, z, l, m, n$ operators which in general do not commute.

Laser frequency noise terms should vanish for arbitrary $C_i$ if:

$$\begin{aligned} p_1 - q_1 + q_2 x - p_3 n &= 0 \\ p_2 - q_2 + q_3 y - p_1 l &= 0 \\ p_3 - q_3 + q_1 z - p_2 m &= 0 \end{aligned}$$

**Solutions:** Polynomial vectors: **module of syzygies**

# The solution space: A module

Eliminate $p_1$ and $p_2$ by Gaussian elimination:

$$\psi(x,y,z;l,m,n) \equiv p_3(1-nlm)+q_1(z-lm)+q_2(xl-1)m+q_3(ym-1)=0$$

$\mathcal{Q}(x,y,z,l,m,n) \equiv \mathcal{K}$ is in general non-commutative polynomial ring.

Consider the map $\varphi : \mathcal{K}^4 \longrightarrow \mathcal{K}$ which takes the polynomial vector $(p_3, q_1, q_2, q_3) \in \mathcal{K}^4$ to $\psi(x,y,z,l,m,n) \in \mathcal{K}$.

$\varphi$ is a homomorphism of modules and its kernel $\varphi^{-1}(0)$ is essentially the submodule we want.

$\varphi$ can be easily extended to $\mathcal{K}^6$ via the elimination equations and then the kernel of this homomorphism is the module $\mathcal{M} \subset \mathcal{K}^6$.

# Ist and IInd generation TDI

- Stationary LISA in flat spacetime: **Ist generation TDI**
  $x = l, y = m, z = n$ - Operators commute

- TDI needs to be generalised taking into consideration, moving LISA, changing armlengths, gravitational field: A **general relativistic** model of LISA optical links is required to account for Sagnac effect, changing armlengths, gravitational field - Shapiro delay, etc.

- Sagnac Effect - **Modified Ist generation TDI**
  Light travel times in two directions around the Sagnac circuit are different:
  $L_i \neq L'_i, \ \Delta(L_1 + L_2 + L_3) \sim 4A\omega/c \sim 28$ km
  Armlengths constant in time, but $x, y, z, l, m, n$ all different - Operators commute

- Time dependent armlengths - **IInd generation TDI**
  $\dot{L}_i \lesssim 10$ m/sec.

$$\phi(t - L_1 - L_2(t - L_1)) \neq \phi(t - L_2 - L_1(t - L_2))$$

For example: $xy \neq yx$ - Operators do not commute

# Complete solution for Ist generation TDI

Some useful but simple examples found by Estabrook, Tinto, Armstrong, JPL, Pasadena, U.S. - adhoc methods used (1999 - 2001).

Symmetric Sagnac:

$$\zeta = yV^1 + zV^2 + xV^3 + yU^1 + zU^2 + xU^3$$

$$yC_1 - xyC_2 + zC_2 - yzC_3 + xC_3 - zxC_1 + yzC_3 - yC_1 + zxC_1 - zC_2 + xyC_2 - xC_3 = 0$$

This is an identity!

Polynomial vectors in the delay operators:

$$\begin{aligned} \zeta &= (y, z, x, y, z, x) \\ X &= (1 - z^2, 0, z(x^2 - 1), 1 - x^2, x(z^2 - 1), 0) \end{aligned}$$

$X$: Michelson - Sensitivity curve of LISA

General method to generate **ALL** Ist generation TDI

# Approximate symmetries and the quotient ring

We drop $\ddot{L} \sim 10^{-6}$ m/sec$^2$ and $\dot{L}^2 \sim 10^{-15}$ terms:

$$(jk - kj)C(t) = (L_j \dot{L}_k - L_k \dot{L}_j)\dot{C}(t - L_j - L_k)$$

If $y_1 y_2 \ldots y_n$ is a permutation of $x_1 x_2 \ldots x_n$, where $x_k$ is any of the operators, then,

$$[x_1 x_2 \ldots x_n, y_1 y_2 \ldots y_n] \simeq 0, \quad n \geq 2$$

For example:

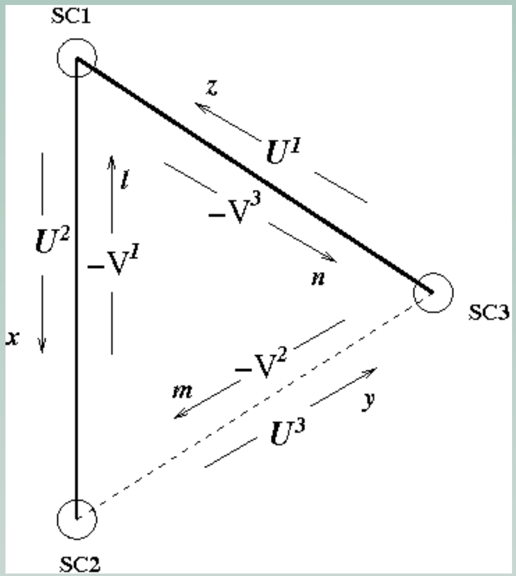
$$[xy, yx] = xyyx - yxxy = xy^2x - yx^2y \simeq 0$$

- Use the (almost) vanishing commutators to generate the left ideal $\mathcal{U}$ - consider all linear combinations of the commutators.
- Form the quotient ring $\bar{\mathcal{K}} \equiv \mathcal{K}/\mathcal{U}$ - any element of $\bar{p} \in \bar{\mathcal{K}}$ is of the form $\bar{p} = p + \mathcal{U}$, where $p \in \mathcal{K}$ - it is an equivalence class of polynomials in $\mathcal{K}$.

Effectively, there are now less polynomials

# IInd generation TDI for LISA: one arm dysfunctional

The general TDI problem extremely difficult: Special case when one of the arms is dysfunctional, say, connecting S/C 2 to S/C 3.




Links: $x, l, z, n$
Polynomials: $p_2 = q_3 = 0$
Reduced equations: $q_2 = -p_1 l,\ p_3 = -q_1 z$
Only one non-trivial equation: $p_1(1 - lx) - q_1(1 - zn) = 0$

# The Michelson solution: round trip TDI operators

Round trip operators: $a = lx, \;\; b = zn$

Equation: $p_1(1-a) - q_1(1-b) = 0$

Michelson solution: $p_1 = 1 - b - ba + ab^2, \;\; q_1 = 1 - a - ab + ba^2$

Solution because $p_1(1-a) - q_1(1-b) \equiv \Delta = [ba, ab] \simeq 0$
Tinto, Armstrong, Estabrook, Vallisneri

Many more solutions possible of higher degrees

Construct commutators $\in \mathcal{U}$ of higher degrees: To each such commutator there is a solution

# Higher degree solutions $n = 2$

Commutator $\Delta = [ab^2a, ba^2b] \simeq 0$

Solution with this commutator $[ab^2a, ba^2b]$:

$$p_1 = 1 - b - ba + ab^2 - ba^2b + ab^2ab + ab^2aba - ba^2bab^2$$

$$q_1 = 1 - a - ab + ba^2 - ab^2a + ba^2ba + ba^2bab - ab^2aba^2$$

Two other commutators: $[a^2b^2, b^2a^2]$, $[abab, baba] \in \mathcal{U}$

These produce two other linearly independent solutions of degree $7$ in $a$, $b$ which is $14$ in $x$, $l$, $z$, $n$.

$p_3$, $q_2$ are of degree $15$.

# Solutions of degree $n \geq 3$

First list the commutators in some order: Choose length lexicographical order $a < b$

$a < b < aa < ab < ba < bb < aaa < aab < aba < abb < baa < bab < bba < bbb < aaaa\ldots.$

A commutator must have equal numbers of $a$'s and $b$'s. List commutators by their first string.

$n = 3$: $aaabbb < aababb < aabbab < aabbba < abaabb\ldots.$ 10 such commutators.

$n = 4$: Length of string 8 in $a, b$ and 35 such commutators

Mathematically: infinite family; Physically restricted by $\ddot{L}$ terms: $n \leq 10$.

# Summary

- **Rings and modules are the underlying mathematical structures for Time-Delay Interferometry.**
- Problem solved for Ist generation TDI and modified Ist generation TDI: problem can be solved if the TDI operators commute.
- For the general problem of IInd generation TDI, the algebraic problem formulated.
- Use symmetries to simplify the IInd generation TDI problem - problem not fully non-commutative - thus simplify the polynomial ring by quotienting it with an ideal generated by near vanishing commutators.
- Family of solutions: IInd generation TDI for the special case when LISA's one arm is dysfunctional.