Numerical calculations of parametric instabilities (PI) in Advanced LIGO interferometer

S. E. Strigin and S. P. Vyatchanin

Moscow State University, Russia, Moscow

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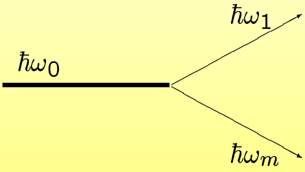
Scheme of energy transformation from main optical mode into the Stokes mode and elastic mode



$$\hbar\omega_0 = \hbar\omega_1 + \hbar\omega_m$$

Two main reasons of carefull calculation of elastic modes

1. The parametric instability condition for signal-recycled LIGO interferometer with optically identical Fabry-Perot cavities[1]:

$$\frac{\Lambda W \omega_1}{2cmL\omega_m\gamma_m}\left(\frac{1/\gamma_+}{1+\Delta^2/\gamma_+^2}+\frac{1/\gamma_-}{1+\Delta^2/\gamma_-^2}\right)>1.$$

The less the detuning $\Delta = \omega_0 - \omega_1 - \omega_m$, the greater the possibility of PI.

[1] A. Gurkovsky, S. E. Strigin and S. P. Vyatchanin, Physics Letters A 362, 91-99,2007.

Two main reasons of carefull calculation of elastic modes

Error of elastic frequency calculations $\Delta\omega_m$ must satisfy $\Delta\omega_m/\gamma_+ \rightarrow \min$ (at least ~ 0.1) where the γ_+ varies in range $1..10\text{sec}^{-1}$. It is extremely necessary!

Elastic mode frequencies help to find the parametric instability precursors and examine them in detail.

Two main reasons of carefull calculation of elastic modes

2. Displacement vector distribution of elastic modes allows to estimate the overlap factor for optical and elastic modes

$$\Lambda = \frac{V\left(\int f_0(\vec{r}) f_1(\vec{r}) u_z d\vec{r}_\perp\right)^2}{\int |f_0|^2 d\vec{r}_\perp \int |f_1|^2 d\vec{r}_\perp \int |\vec{u}|^2 dV}$$

A lot of combinations of elastic and optical modes have an overlap factor greater than $\Lambda > 10^{-4}$ and it has been calculated in papers [2].

Overlap factor error in the calculations(at least 1..10%) is less "dangerous"than the frequency error but is has to be taken into account too.

[2] L.Ju, S. Gras, C. Zhao, J. Degallaix, and D.G.Blair, Phys.Lett.A, 354, 360-365, 2006;

L.Ju, C. Zhao, S. Gras, J. Degallaix, D.G.Blair, J. Munch, D.H. Reitze, Phys.Lett.A, 355, 419-426, 2006.

Previous calculations of elastic modes

- The accuracy of FEMLAB code is $1 \div 3\%$ in the frequency range $4.4 \div 27$kHz(for fused silica)
  V. B. Braginsky, S. E. Strigin and S. P. Vyatchanin, Physics Letters A305, 111-124, 2002.
- With very high mesh density the accuracy of calculating of elastic mode frequencies with ANSYS code is within 0.1% at 50kHz(greater than cavity bandwidth) with 60000 meshing elements(both for fused silica and sapphire)
  C.Zhao, L.Ju, J. Degallaix, S.Gras, D.G.Blair, Phys.Rev.Lett., 94, 121102, 2005.
  L.Ju, S. Gras, C. Zhao, J. Degallaix, and D.G.Blair, Phys.Lett.A, 354, 360-365, 2006; L.Ju, C. Zhao, S. Gras, J. Degallaix, D.G.Blair, J. Munch, D.H. Reitze, Phys.Lett.A, 355, 419-426, 2006.

How to test the accuracy of COMSOL program?

The accuracy of numerical calculations can be estimated comparing difference between frequencies calculated on meshes with different but increased number of nodes.

How to test the accuracy of COMSOL program?

Chree-Lamb axial-symmetric elastic modes with simple analytical form for displacement vector components and frequency[3]:

$$u_r = -CJ_1(\beta r)\cos(\beta z),\; u_z = CJ_0(\beta r)\sin(\beta z),\;\; u_\phi = 0,$$

$$\omega = \sqrt{2}\beta\sqrt{E/2\rho(1+\sigma)},$$

where the parameter β satisfies:

$$J_1'(\beta R) = 0,\;\; \cos(\beta H/2) = 0,$$

i.e. these modes exist for specific ratio R/H.
These modes are partial solutions of elastic equation, but they are the important "beacons"to obtain the accuracy of COMSOL code!

[3] Lamb H., Proc. Roy. Soc. Lond. A93, 648, 114-121, 1917;
Chree C., Quart. J. Pure and Appl. Math.,21, 83/84, 287-298,1886.

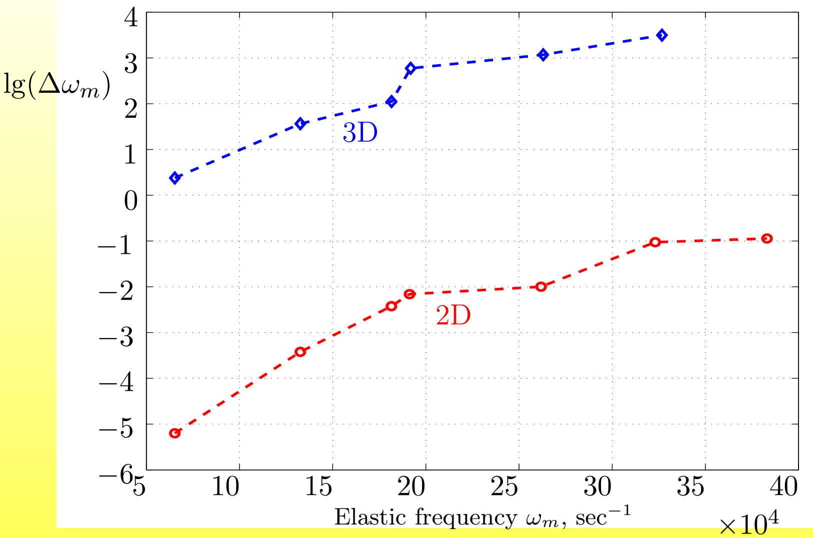
Two built-in modules in the COMSOL code

- 2D built-in module: there is no dependence on azimuth angle ϕ, all elastic modes are axial symmetric ones
- 3D built-in module: it finds all the elastic solutions (axial symmetric and non-axial symmetric with dependence on $e^{im\phi}$ for any azimuth number m) and eigenfrequencies

Calculations of Chree-Lamb modes

- Chree-Lamb modes exist for specific ratio R/H
- We calculate the elastic Chree-Lamb frequencies for such ratios R/H which are close to ratio for Advanced LIGO interferometer $R/H \simeq 1.26$ with fixed mirror volume as LIGO's mirror
- We estimate the error of numerical calculations $\Delta\omega_m/\omega_m$

COMSOL's accuracy achieved for elastic Chree-Lamb modes



The graph has been calculated for fused silica mirrors with fixed volume V as LIGO 40kg mirror

Calculations of elastic modes in 2D and 3D modules in frequency range from $2\pi \times 10^4 \mathrm{sec}^{-1}$ to $2\pi \times 6.5 \times 10^4 \mathrm{sec}^{-1}$

- The 2D module accuracy $\Delta\omega_m/\omega_m$ of calculated elastic modes varies in range $10^{-9} \div 10^{-7}$ in this frequency range;
- Value $\Delta\omega_m/\gamma_+$ in 2D module varies in range $6.3 \times 10^{-5} \div 0.1$ if $\gamma_+ \simeq 1 \mathrm{sec}^{-1}$;

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- The 3D module accuracy $\Delta\omega_m/\omega_m$ of calculated elastic modes varies in range $10^{-5} \div 10^{-2}$;
- Value $\Delta\omega_m/\gamma_+$ in 3D module varies in range $0.7 \div 3000$ if $\gamma_+ \simeq 1\text{sec}^{-1}$(it is a poor accuracy).

Calculation of non-axial symmetric elastic modes

If we consider the components of displacement vector in form

$$u_r(r,\phi,z) = u_r(r,z)e^{im\phi}, \;\; u_z(r,\phi,z) = u_z(r,z)e^{im\phi},$$

$$u_\phi(r,\phi,z) = iu_\phi(r,z)e^{im\phi},$$

we can rewrite elastic equation and boundary conditions into 2D space with two independent coordinates r, z and solve it using 2D module which has a high numerical accuracy.

Frequencies of elastic modes with mirror's pins

Pins that mirrors are suspended on slightly change the frequencies of elastic modes.

Rough estimation of elastic frequency shift[4]:

$$\frac{\Delta\omega_m}{\omega_m} \simeq \frac{2V_{\rm pin}}{V} \times 2 = \frac{2m_{\rm pin}}{m} \times 2 \simeq 1 \times 10^{-3}!$$

This value is comparable with the COMSOL achieved accuracy for 3D module.

[4] V. B. Braginsky, S. E. Strigin and S. P. Vyatchanin, Physics Letters A305, 111-124, 2002.

Change in frequency structure of elastic modes with mirror's pins

- Mirrors without pins(ideal cylinder): Elastic modes with same frequencies (degenerate mode case) have different dependence on azimuth angle ϕ ($\sim \cos(m\phi)$ and $\sin(m\phi)$);
- Mirrors with pins: These modes have slightly different frequencies (doublets) and different detunings Δ so that each of them can be the candidate for PI.

More accurate way for estimation of accuracy $\Delta\omega_m/\omega_m$ for elastic mode frequencies with mirror's pins

The average kinetic and potential energies of elastic deformations are equal to each other for mirrors without pins:

$$\frac{1}{2}\rho\omega_m^2\int_V u_i^2\,dV = \frac{1}{2}\int_V u_{ij}\sigma_{ij}\,dV$$

and in the case with pins:

$$\frac{1}{2}\rho(\omega_m+\Delta\omega_m)^2\int_{V+\Delta V} u_i^2\,dV = \frac{1}{2}\int_{V+\Delta V} u_{ij}\sigma_{ij}\,dV.$$

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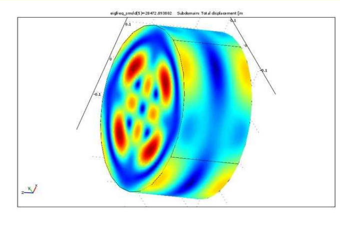
Combining these formulas and assuming that $\Delta\omega_m \ll \omega_m$ we obtain:

$$\frac{\Delta\omega_m}{\omega_m} \simeq \frac{\int_{\Delta V} u_{ij}\sigma_{ij}\,dV - \rho\omega_m^2\int_{\Delta V} u_i^2\,dV}{2\rho\omega_m^2\int_V u_i^2\,dV}.$$

Doublets frequency shifts



- For parametric unstable elastic mode[5] with frequency $\omega_m = 2\pi \times 2.84 \times 10^4 \mathrm{sec}^{-1}$(3D module accuracy $\Delta\omega_m \simeq 100\mathrm{sec}^{-1}, m = 2$)we can estimate the frequency shifts of the doublets in case with pins situated on the lateral surface of the mirror in the opposite directions centered at z-coordinate $H/2$:

$$\Delta\omega_{m1} \simeq -0.5\mathrm{sec}^{-1},\ \Delta\omega_{m2} \simeq -26.5\mathrm{sec}^{-1}.$$

[5] L.Ju, S. Gras, C. Zhao, J. Degallaix, and D.G.Blair, Phys.Lett.A, 354, 360-365, 2006

Doublets frequency shifts

- The frequency difference between these elastic doublets for the mirrors with pins is large enough: for elastic frequency $\omega_m = 2\pi \times 2.84 \times 10^4 \mathrm{sec}^{-1}$ it is equal $\Delta\omega_{12} \simeq 26 \mathrm{sec}^{-1}$.
- This difference is greater than the relaxation rates of symmetrical and anti-symmetrical modes $\gamma_+,\ \gamma_- \geq 1.5 \mathrm{sec}^{-1}$ and each doublet can contribute into the parametric instability separately.

Conclusions

- Using analytical solutions for Chree-Lamb elastic modes as a beacons we can effectively estimate error of numerical calculation of elastic modes;
- This error $\Delta\omega_m$ has to be less than relaxation rate γ_+ of Stokes mode.

We have to know our "enemy"in detail in order to destroy it!

Plans for future research

- It is attractive to transform 3D problem into 2D problem introducing obvious dependence of all displacement components on azimuth angle ϕ as $\sim e^{im\phi}$;
- We have to estimate shifts and splitting of each elastic modes in test mass with pins or other imperfections.

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